

# Four-dimensional almost Hermitian manifolds with vanishing Tricerri-Vanhecke Bochner curvature tensor

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## Abstract

We study curvature properties of four-dimensional almost Hermitian manifolds with vanishing Bochner curvature tensor as defined by Tricerri and Vanhecke. We give local structure theorems for such Kähler manifolds, and find out several examples related to the theorems.

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## 1 Introduction

The Bochner curvature tensor  $B$  was defined by Bochner as a formal analogy of the Weyl conformal curvature tensor [2]. The Bochner Kähler manifold which is a Kähler manifold with vanishing Bochner curvature tensor has been studied by Kamishima [9] and Bryant [3]. Tricerri and Vanhecke [20] studied the decomposition of the space of all curvature tensors on a Hermitian vector space from the view-point of unitary representation theory and defined a Bochner type conformal curvature tensor  $B(R)$  for any almost Hermitian manifold  $M = (M, J, g)$ . Then tensor field  $B(R)$  is invariant under conformal change of the Riemannian metric  $g$ . On one hand, Matsuo [13] introduced a generalization of the Bochner curvature tensor which is called the pseudo-Bochner curvature tensor on a Hermitian manifold  $M = (M, J, g)$  and denoted with  $B_H$ , and discussed several curvature properties.

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In the present paper, we shall study the curvature properties of four-dimensional almost Hermitian manifolds with vanishing Tricerri-Vanhecke Bochner curvature tensor. In the sequel, we shall call an almost Hermitian manifold with vanishing Tricerri-Vanhecke Bochner curvature tensor a Tricerri-Vanhecke Bochner flat one, and also call a four-dimensional almost Hermitian manifold an almost Hermitian surface.

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## 2 Preliminaries

Let  $M = (M, J, g)$  be a  $2n$ -dimensional almost Hermitian manifold and  $\Omega$  the Kähler form of  $M$  defined by  $\Omega(X, Y) = g(JX, Y)$ , for  $X, Y \in \mathfrak{X}(M)$ , where  $\mathfrak{X}(M)$  denotes the Lie algebra of all smooth vector fields  $X, Y$  on  $M$ . We denote by  $\nabla$  and  $R$  the Levi-Civita connection and the curvature tensor of  $(M, J, g)$  defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \quad (2.1)$$

for  $X, Y, Z \in \mathfrak{X}(M)$ . Further, we denote by  $\rho$ ,  $\rho^*$ ,  $\tau$  and  $\tau^*$  the Ricci tensor, the Ricci  $*$ -tensor, the scalar curvature and the  $*$ -scalar curvature defined respectively as:

$$\begin{aligned} \rho(X, Y) &= \text{tr} (Z \mapsto R(Z, X)Y), \\ \rho^*(X, Y) &= \text{tr} (Z \mapsto R(X, JZ)JY), \\ \tau &= \text{tr } Q, \quad \tau^* = \text{tr } Q^* \end{aligned} \quad (2.2)$$

where  $Q$  and  $Q^*$  are the Ricci operator and the Ricci  $*$ -operator defined by  $g(QX, Y) = \rho(X, Y)$  and  $g(Q^*X, Y) = \rho^*(X, Y)$ , for  $X, Y \in \mathfrak{X}(M)$ , respectively. We may easily check that  $\rho^*(X, Y) = \rho^*(JY, JX)$  holds for all  $X, Y \in \mathfrak{X}(M)$ , and  $\rho^* = \rho$  holds if  $M$  is a Kähler manifold. An almost Hermitian manifold  $M$  is called a *weakly  $*$ -Einstein* manifold if  $\rho^* = \frac{\tau^*}{2n}g$  holds on  $M$  and also called a  *$*$ -Einstein* manifold especially if  $\tau^*$  is constant. We denote by  $\mathcal{R}$  the curvature operator defined by

$$g(\mathcal{R}(\iota(x) \wedge \iota(y)), \iota(z) \wedge \iota(w)) = -g(R(x, y)z, w) = -R(x, y, z, w), \quad (2.3)$$

for  $x, y, z, w \in T_p M$ ,  $p \in M$ , where  $\iota$  denotes the duality :  $TM \longrightarrow \wedge^1 M = T^*M$  (the cotangent bundle of  $M$ ). Let  $\{e_i\}$  be an orthonormal basis of  $T_p M$  at any point  $p \in M$ .

In this paper, we shall adopt the following notational convention:

$$\begin{aligned}
R_{ijkl} &= g(R(e_i, e_j)e_k, e_l), \\
R_{\bar{i} jkl} &= g(R(Je_i, e_j)e_k, e_l), \\
&\dots\dots \\
R_{\bar{i} \bar{j} \bar{k} \bar{l}} &= g(R(Je_i, Je_j)Je_k, Je_l), \\
\rho_{ij} &= \rho(e_i, e_j), \quad \dots, \quad \rho_{\bar{i} \bar{j}} = \rho(Je_i, Je_j), \\
\rho_{ij}^* &= \rho^*(e_i, e_j), \quad \dots, \quad \rho_{\bar{i} \bar{j}}^* = \rho^*(Je_i, Je_j), \\
J_{ij} &= g(Je_i, e_j), \quad \nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k),
\end{aligned} \tag{2.4}$$

and so on, where the Latin indices run over the range  $1, 2, \dots, 2n$ .

The Bochner curvature tensor  $B(R)$  defined by Tricerri and Vanhecke is stated below:

$$\begin{aligned}
B(R) = & R - \frac{1}{4(n+2)(n-2)} g \triangle \rho + \frac{2n-3}{4(n-1)(n-2)} g \otimes \rho \\
& - \frac{1}{4(n+2)(n-2)} g \triangle (\rho J) + \frac{1}{4(n-1)(n-2)} g \otimes (\rho J) \\
& + \frac{2n^2-5}{4(n+1)(n+2)(n-2)} g \triangle \rho^* - \frac{2n-1}{4(n+1)(n-2)} g \otimes \rho^* \\
& + \frac{3}{4(n+1)(n+2)(n-2)} g \triangle (\rho^* J) - \frac{3}{4(n+1)(n-2)} g \otimes (\rho^* J) \\
& + \frac{3n\tau - (2n^2 - 3n + 4)\tau^*}{16(n+1)(n+2)(n-1)(n-2)} g \triangle g - \frac{\tau - \tau^*}{8(n-1)(n-2)} g \otimes g
\end{aligned} \tag{2.5}$$

for  $n \geq 3$ , and

$$\begin{aligned}
B(R) = & R + \frac{1}{2} g \otimes \rho + \frac{1}{12} \{ g \triangle \rho^* - g \otimes \rho^* - g \triangle (\rho^* J) + g \otimes (\rho^* J) \} \\
& + \frac{3\tau^* - \tau}{96} g \triangle g - \frac{\tau + \tau^*}{16} g \otimes g
\end{aligned} \tag{2.6}$$

for  $n = 2$ , where for any  $(0,2)$ -tensors  $a$  and  $b$ , we set

$$\begin{aligned}
& (a \otimes b)(x, y, z, w) \\
& = a(x, z)b(y, w) - a(x, w)b(y, z) + b(x, z)a(y, w) - b(x, w)a(y, z),
\end{aligned} \tag{2.7}$$

$$\bar{a}(x, y) = a(x, Jy), \tag{2.8}$$

for  $x, y, z, w \in T_p M$ ,  $p \in M$ , and we set

$$a \triangle b = a \otimes b + \bar{a} \otimes \bar{b} + 2\bar{a} \otimes \bar{b} + 2\bar{b} \otimes \bar{a}. \tag{2.9}$$

Further, the Weyl curvature tensor is given by

$$W = R + \frac{1}{2n-2} g \otimes \rho - \frac{\tau}{2(2n-1)(2n-2)} g \otimes g. \quad (2.10)$$

We denote by  $\mathcal{W}$  the Weyl curvature operator.

We note that the Tricerri-Vanhecke Bochner curvature tensor  $B(R)$  coincides with the usual Bochner curvature tensor  $B$  on Kähler manifold [20].

### 3 Local structures of Tricerri-Vanhecke Bochner flat almost Hermitian surfaces

In this section, we shall discuss Tricerri-Vanhecke Bochner flat almost Hermitian surfaces and give some local structure theorems for these surfaces. Let  $M = (M, J, g)$  be a Tricerri-Vanhecke Bochner flat almost Hermitian surface. Then, by (2.6), the curvature tensor  $R$  of  $M$  can be expressed explicitly by

$$\begin{aligned} R(X, Y, Z, W) &= \frac{1}{2} \left\{ g(X, W)\rho(Y, Z) + g(Y, Z)\rho(X, W) \right. \\ &\quad \left. - g(X, Z)\rho(Y, W) - g(Y, W)\rho(X, Z) \right\} \\ &\quad + \frac{1}{12} \left\{ 2g(X, JY) \left( \rho^*(W, JZ) - \rho^*(JZ, W) \right) \right. \\ &\quad \quad + 2g(Z, JW) \left( \rho^*(Y, JX) - \rho^*(JX, Y) \right) \\ &\quad \quad + g(X, JZ) \left( \rho^*(W, JY) - \rho^*(JY, W) \right) \\ &\quad \quad + g(Y, JW) \left( \rho^*(Z, JX) - \rho^*(JX, Z) \right) \\ &\quad \quad + g(X, JW) \left( \rho^*(Y, JZ) - \rho^*(JZ, Y) \right) \\ &\quad \quad \left. + g(Y, JZ) \left( \rho^*(X, JW) - \rho^*(JW, X) \right) \right\} \\ &\quad + \frac{3\tau^* - \tau}{48} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right. \\ &\quad \quad - 2g(X, JY)g(Z, JW) - g(X, JZ)g(Y, JW) \\ &\quad \quad \left. + g(Y, JZ)g(X, JW) \right\} \\ &\quad - \frac{\tau + \tau^*}{8} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right\} \end{aligned} \quad (3.1)$$

for  $X, Y, Z, W \in \mathfrak{X}(M)$ . On one hand, from (2.10), the Weyl curvature tensor  $W$  is given by

$$\begin{aligned}
& W(X, Y, Z, W) \\
&= R(X, Y, Z, W) \\
&\quad - \frac{1}{2} \left\{ g(X, W) \rho(Y, Z) + g(Y, Z) \rho(X, W) \right. \\
&\quad \left. - g(X, Z) \rho(Y, W) - g(Y, W) \rho(X, Z) \right\} \\
&\quad + \frac{\tau}{6} \{ g(X, W) g(Y, Z) - g(X, Z) g(Y, W) \}
\end{aligned} \tag{3.2}$$

for  $X, Y, Z, W \in \mathfrak{X}(M)$ . From (3.1) and (3.2), the Weyl curvature tensor  $W$  is also expressed by

$$\begin{aligned}
& W(X, Y, Z, W) \\
&= \frac{\tau - 3\tau^*}{24} \left\{ g(X, W) g(Y, Z) - g(X, Z) g(Y, W) \right\} \\
&\quad + \frac{1}{12} \left\{ 2g(X, JY) \left( \rho^*(W, JZ) - \rho^*(JZ, W) \right) \right. \\
&\quad \quad + 2g(Z, JW) \left( \rho^*(Y, JX) - \rho^*(JX, Y) \right) \\
&\quad \quad + g(X, JZ) \left( \rho^*(W, JY) - \rho^*(JY, W) \right) \\
&\quad \quad + g(Y, JW) \left( \rho^*(Z, JX) - \rho^*(JX, Z) \right) \\
&\quad \quad + g(X, JW) \left( \rho^*(Y, JZ) - \rho^*(JZ, Y) \right) \\
&\quad \quad \left. + g(Y, JZ) \left( \rho^*(X, JW) - \rho^*(JW, X) \right) \right\} \\
&\quad + \frac{3\tau^* - \tau}{48} \left\{ g(X, W) g(Y, Z) - g(X, Z) g(Y, W) \right. \\
&\quad \quad - 2g(X, JY) g(Z, JW) - g(X, JZ) g(Y, JW) \\
&\quad \quad \left. + g(Y, JZ) g(X, JW) \right\}
\end{aligned} \tag{3.3}$$

for  $X, Y, Z, W \in \mathfrak{X}(M)$ . First, from (3.1), by direct calculation, we have the following theorem.

**Theorem 3.1** *Let  $M = (M, J, g)$  be a Tricerri-Vanhecke Bochner flat almost Hermitian surface. Then, the curvature tensor  $R$  satisfies the following curvature identity*

$$\begin{aligned}
& R(X, Y, Z, W) - R(JX, JY, Z, W) \\
&\quad - R(X, Y, JZ, JW) + R(JX, JY, JZ, JW) \\
&= R(X, JY, Z, JW) + R(X, JY, JZ, W) \\
&\quad + R(JX, Y, JZ, W) + R(JX, Y, Z, JW)
\end{aligned} \tag{3.4}$$

for  $X, Y, Z, W \in \mathfrak{X}(M)$ .

**Proof** From (3.1), the left-hand side of (3.4) is

$$\begin{aligned}
& R(X, Y, Z, W) - R(JX, JY, Z, W) \\
& \quad - R(X, Y, JZ, JW) + R(JX, JY, JZ, JW) \\
& = \frac{1}{2} \left\{ g(X, W)\rho(Y, Z) + g(Y, Z)\rho(X, W) \right. \\
& \quad - g(X, Z)\rho(Y, W) - g(Y, W)\rho(X, Z) \\
& \quad - g(X, JW)\rho(Y, JZ) - g(Y, JZ)\rho(X, JW) \\
& \quad + g(X, JZ)\rho(Y, JW) + g(Y, JW)\rho(X, JZ) \\
& \quad + g(X, W)\rho(JY, JZ) + g(Y, Z)\rho(JX, JW) \\
& \quad - g(X, Z)\rho(JY, JW) - g(Y, W)\rho(JX, JZ) \\
& \quad + g(X, JW)\rho(JY, Z) + g(Y, JZ)\rho(JX, W) \\
& \quad - g(X, JZ)\rho(JY, W) - g(Y, JW)\rho(JX, Z) \left. \right\} \\
& \quad - \frac{\tau + \tau^*}{4} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right. \\
& \quad \left. - g(X, JW)g(Y, JZ) + g(X, JZ)g(Y, JW) \right\}
\end{aligned} \tag{3.5}$$

and the right-hand side of (3.4) is

$$\begin{aligned}
& R(X, JY, Z, JW) + R(X, JY, JZ, W) \\
& \quad + R(JX, Y, JZ, W) + R(JX, Y, Z, JW) \\
& = \frac{1}{2} \left\{ g(X, W)\rho(Y, Z) + g(Y, Z)\rho(X, W) \right. \\
& \quad - g(X, Z)\rho(Y, W) - g(Y, W)\rho(X, Z) \\
& \quad - g(X, JW)\rho(Y, JZ) - g(Y, JZ)\rho(X, JW) \\
& \quad + g(X, JZ)\rho(Y, JW) + g(Y, JW)\rho(X, JZ) \\
& \quad + g(X, W)\rho(JY, JZ) + g(Y, Z)\rho(JX, JW) \\
& \quad - g(X, Z)\rho(JY, JW) - g(Y, W)\rho(JX, JZ) \\
& \quad + g(X, JW)\rho(JY, Z) + g(Y, JZ)\rho(JX, W) \\
& \quad - g(X, JZ)\rho(JY, W) - g(Y, JW)\rho(JX, Z) \left. \right\} \\
& \quad - \frac{\tau + \tau^*}{4} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right. \\
& \quad \left. - g(X, JW)g(Y, JZ) + g(X, JZ)g(Y, JW) \right\}.
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we may see that the curvature identity (3.4) holds.  $\square$

**Remark** It is known that the curvature tensor of any Hermitian manifold satisfies the curvature identity (3.4) in the above Theorem 3.1 [6]. However, the converse is not true in general. In fact, Tricerri and Vanhecke [19] gave an example of a locally flat almost Hermitian surface which is not Hermitian.

Now, let  $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$  be a unitary basis (resp. any local unitary frame field) of  $T_p M$  for any  $p \in M$ , and  $\{e^i\}$  be the dual basis (resp. local dual unitary frame field) of  $\{e_i\}$ . The space  $\wedge_p^2 M$  of all 2-forms on  $M$  is decomposed by

$$\wedge_p^2 M = \wedge_+^2 \oplus \wedge_-^2, \quad (3.7)$$

and these subspaces are spanned respectively by

$$\wedge_+^2 = \text{span} \{ \Omega_0, \Phi, J\Phi \}, \quad \wedge_-^2 = \text{span} \{ \Psi_1, \Psi_2, \Psi_3 \}, \quad (3.8)$$

where

$$\begin{aligned} \Omega_0 &= \frac{1}{\sqrt{2}}\Omega = \frac{1}{\sqrt{2}}(e^1 \wedge e^2 + e^3 \wedge e^4), \\ \Phi &= \frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4), \quad J\Phi = \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3), \\ \Psi_1 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^2 - e^3 \wedge e^4), \\ \Psi_2 &= \frac{1}{\sqrt{2}}(e^1 \wedge e^3 + e^2 \wedge e^4), \quad \Psi_3 = \frac{1}{\sqrt{2}}(e^1 \wedge e^4 - e^2 \wedge e^3). \end{aligned} \quad (3.9)$$

Then, from (3.3) and (3.9), we have

$$\begin{aligned} \mathcal{W}(\Omega_0) &= \frac{3\tau^* - \tau}{12}\Omega_0 - \frac{1}{2}(\rho_{14}^* - \rho_{41}^*)\Phi + \frac{1}{2}(\rho_{13}^* - \rho_{31}^*)J\Phi, \\ \mathcal{W}(\Phi) &= -\frac{1}{2}(\rho_{14}^* - \rho_{41}^*)\Omega_0 - \frac{3\tau^* - \tau}{24}\Phi, \\ \mathcal{W}(J\Phi) &= \frac{1}{2}(\rho_{13}^* - \rho_{31}^*)\Omega_0 - \frac{3\tau^* - \tau}{24}J\Phi, \\ \mathcal{W}(\Psi_i) &= 0, \quad (i = 1, 2, 3), \end{aligned} \quad (3.10)$$

where  $\mathcal{W}$  is the Weyl curvature operator. Thus, by (3.8) and (3.10), we have the following theorems.

**Theorem 3.2** *Let  $M = (M, J, g)$  be a Tricerri-Vanhecke Bochner flat almost Hermitian surface. Then,  $M$  is self-dual.*

**Theorem 3.3** *Let  $M = (M, J, g)$  be a Tricerri-Vanhecke Bochner flat almost Hermitian surface. Then,  $M$  is anti-self-dual if and only if  $\rho^*$  is symmetric and  $3\tau^* - \tau = 0$  holds on  $M$ .*

**Corollary 3.4** *Let  $M = (M, J, g)$  be a Tricerri-Vanhecke Bochner flat almost Hermitian surface. Then,  $M$  is conformally flat if and only if  $\rho^*$  is symmetric and  $3\tau^* - \tau = 0$  holds on  $M$ .*

Below are two examples of conformally flat, Tricerri-Vanhecke Bochner flat almost Hermitian surfaces.

**Example 1** Let  $M = \mathbb{R}_+^4 = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 > 0, x_1, x_2, x_3 \in \mathbb{R} \}$  and  $\{e_1, e_2, e_3, e_4\}$  be the global frame field on  $M$  defined by

$$e_1 = x_4 \frac{\partial}{\partial x_1}, \quad e_2 = x_4 \frac{\partial}{\partial x_2}, \quad e_3 = x_4 \frac{\partial}{\partial x_3}, \quad e_4 = x_4 \frac{\partial}{\partial x_4}. \quad (3.11)$$

Further, we define almost Hermitian structure  $(J, g)$  on  $M$  as follows:

$$J : \quad e_1 \longmapsto e_2, \quad e_2 \longmapsto -e_1, \quad e_3 \longmapsto e_4, \quad e_4 \longmapsto -e_3, \quad (3.12)$$

and

$$g(e_i, e_j) = \delta_{ij}. \quad (3.13)$$

Then, we may easily check that  $(M, J, g)$  is a Hermitian surface of constant sectional curvature  $-1$  (and hence, conformally flat, Tricerri-Vanhecke Bochner flat Hermitian surface by virtue of (3.1)).

**Example 2** [18] Let  $M_1 = (M_1(K), J_1, g_1)$ ,  $M_2 = (M_2(-K), J_2, g_2)$  be oriented surfaces with constant Gaussian curvatures  $K$  and  $-K$  ( $K > 0$ ) respectively, and  $(M, J, g) = (M_1 \times M_2, J_1 \times J_2, g_1 \times g_2)$  be the direct product of  $M_1$  and  $M_2$ .

We may immediately observe that a complex space form is a typical example of Tricerri-Vanhecke Bochner flat Kähler manifold and the above Example 2 is such an example. Now, concerning the Example 2, we have the following theorem.

**Theorem 3.5** *Let  $M = (M, J, g)$  be a Tricerri-Vanhecke Bochner flat Kähler surface. If the scalar curvature  $\tau$  of  $M$  is constant, then  $M$  is locally a complex space form of complex dimension 2, or locally a product of two oriented surfaces of different constant Gaussian curvatures  $K$  and  $-K$  ( $K \neq 0$ ).*

**Proof** Let  $\lambda, \mu$  ( $\lambda \geq \mu$ ) be the eigenvalues of the Ricci transformation  $Q$  at each point of  $M$ . Then, we may easily observe that  $\lambda + \mu = \frac{\tau}{2}$  and the eigenvalues  $\lambda, \mu$  give rise to continuous functions on  $M$ . Now, we set  $M_0 = \{p \in M \mid \lambda > \mu \text{ at } p\}$ . Then,  $M_0$  is an open set (possibly, empty set) of  $M$ .

First, we assume that  $M_0$  is empty. Then, we see that  $M$  is Einstein, and hence, by (3.1),  $M$  is locally a complex space form of complex dimension 2 of constant holomorphic sectional curvature  $\frac{\tau}{6}$ .

Next, we assume that  $M_0$  is not empty. Then, we may define two smooth  $J$ -invariant distributions  $D_\lambda$  and  $D_\mu$  on  $M_0$  corresponding to the eigenvalues  $\lambda$  and  $\mu$  of the Ricci transformation  $Q$ . Let  $U$  be an any component of  $M_0$  and  $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$  be any local unitary frame field in  $U$  such that  $Qe_1 = \lambda e_1$  ( $Qe_2 = \lambda e_2$ ),  $Qe_3 = \mu e_3$  ( $Qe_4 = \mu e_4$ ). We set

$$\nabla_{e_i} e_j = \sum_k \Gamma_{ijk} e_k \quad (i, j = 1, 2, 3, 4). \quad (3.14)$$

Then, since  $M$  is Kähler, we get

$$\Gamma_{ijk} = -\Gamma_{ikj}, \quad \Gamma_{i\bar{j}\bar{k}} = \Gamma_{ijk} \quad (i, j, k = 1, 2, 3, 4). \quad (3.15)$$

On one hand, from (3.1), since  $\tau$  is constant, we have

$$\sum_i (\nabla_{e_i} R)(X, Y, Z, e_i) = \frac{1}{2} \{(\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z)\} \quad (3.16)$$

and hence, taking account of the second Bianchi identity,

$$(\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z) = 0 \quad (3.17)$$

for any  $X, Y, Z \in \mathfrak{X}(U)$ . Thus, by setting  $(X, Y, Z) = (e_1, e_2, e_2)$ ,  $(e_1, e_3, e_3)$ ,  $(e_1, e_4, e_4)$  in (3.17), from (3.15), we have respectively

$$e_1 \lambda = 0, \quad (3.18)$$

$$e_1 \mu + \Gamma_{342}(\lambda - \mu) = 0, \quad (3.19)$$

$$e_1 \mu - \Gamma_{432}(\lambda - \mu) = 0. \quad (3.20)$$

From (3.18) and the hypothesis ( $\tau = 2(\lambda + \mu)$  is constant), we have  $e_1 \mu = 0$ . Thus, by (3.19) and (3.20), we have  $\Gamma_{342} = \Gamma_{432} = 0$ . Similarly, we have  $e_a \lambda = e_a \mu = 0$  ( $a = 2, 3, 4$ ), and  $\Gamma_{341} = \Gamma_{431} = \Gamma_{142} = \Gamma_{241} = \Gamma_{132} = \Gamma_{231} = 0$ . Thus, we see that the distributions  $D_\lambda$  and  $D_\mu$  are both parallel ones on each  $U$ . Therefore,  $M_0$  is locally a product of two integral manifolds with respect to the distributions  $D_\lambda$  and  $D_\mu$ . From (3.1),  $0 = R_{1313} = -\frac{\tau}{24}$

and hence  $\tau = 0$ . Since  $\lambda$  and  $\mu$  are both constant, by setting  $\lambda = K$  and  $\mu = -K$  from taking account of  $\lambda + \mu = \frac{\tau}{2} = 0$ , we see that  $M_0 = M$  and  $M$  is locally a product of two oriented surfaces of different constant Gaussian curvatures  $K$  and  $-K$  ( $K \neq 0$ ).  $\square$

We note that Bochner flat Kähler manifold with constant scalar curvature is locally symmetric in any dimension [14].

The following example illustrates the above Theorem 3.5. Namely, there exists a Tricerri-Vanhecke Bochner flat almost Kähler surface with constant scalar curvature which is not a Kähler one.

**Example 3** We set  $(M, g) = \mathbb{H}^3(-1) \times \mathbb{R}$ , where  $\mathbb{H}^3(-1)$  is a 3-dimensional real hyperbolic space of constant sectional curvature  $-1$  and  $\mathbb{R}$  is a real line. Let

$$e_1 = x_1 \frac{\partial}{\partial x_1}, \quad e_2 = x_1 \frac{\partial}{\partial x_2}, \quad e_3 = x_1 \frac{\partial}{\partial x_3}, \quad e_4 = \frac{\partial}{\partial x_4}.$$

on  $M = \mathbb{R}_+^4 = \mathbb{R}_+^3 \times \mathbb{R} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 > 0\}$  and define an almost Hermitian structure  $(J, g)$  on  $M$  by  $g(e_i, e_j) = \delta_{ij}$  and  $Je_i = \sum_{j=1}^4 J_{ij}e_j$ , where

$$(J_{ij}) = \begin{pmatrix} 0 & \cos x_4 & \sin x_4 & 0 \\ -\cos x_4 & 0 & 0 & -\sin x_4 \\ -\sin x_4 & 0 & 0 & \cos x_4 \\ 0 & \sin x_4 & -\cos x_4 & 0 \end{pmatrix}.$$

We denote by  $\{e^i\}_{i=1, \dots, 4}$  the dual basis of  $\{e_i\}$ . Then the Kähler form  $\Omega$  is given by

$$\begin{aligned} \Omega &= J_{12}e^1 \wedge e^2 + J_{13}e^1 \wedge e^3 + J_{14}e^1 \wedge e^4 \\ &\quad + J_{23}e^2 \wedge e^3 + J_{24}e^2 \wedge e^4 + J_{34}e^3 \wedge e^4 \\ &= \frac{1}{x_1^2} \cos x_4 dx_1 \wedge dx_2 + \frac{1}{x_1^2} \sin x_4 dx_1 \wedge dx_3 \\ &\quad - \frac{1}{x_1} \sin x_4 dx_2 \wedge dx_4 + \frac{1}{x_1} \cos x_4 dx_3 \wedge dx_4 \end{aligned} \tag{3.21}$$

Thus, we have  $d\Omega = 0$ , and hence  $(M, J, g)$  is an almost Kähler manifold.

We may easily check that Example 3 is a locally symmetric, conformally flat, Tricerri-Vanhecke Bochner flat, non-Kähler, almost Kähler surface with constant scalar curvature  $\tau = -6$  and constant  $\ast$ -scalar curvature  $\tau^* = -2$ .

## 4 Compact Tricerri-Vanhecke Bochner flat almost Hermitian surfaces

Let  $M = (M, J, g)$  be a compact Tricerri-Vanhecke Bochner flat almost Hermitian surface. From (3.10), we have

$$\begin{aligned}\|\mathcal{W}_+\|^2 &= \frac{(3\tau^* - \tau)^2}{96} + \frac{1}{2} \{(\rho_{13}^* - \rho_{31}^*)^2 + (\rho_{14}^* - \rho_{41}^*)^2\}, \\ \|\mathcal{W}_-\|^2 &= 0.\end{aligned}\quad (4.1)$$

We set

$$G = \sum_{i,j} (\rho_{ij}^* - \rho_{ji}^*)^2 = 4 \{(\rho_{13}^* - \rho_{31}^*)^2 + (\rho_{14}^* - \rho_{41}^*)^2\}. \quad (4.2)$$

From (4.1), taking account of (4.2), the first Pontrjagin number is given by

$$\begin{aligned}p_1(M) &= \frac{1}{4\pi^2} \int_M \{\|\mathcal{W}_+\|^2 - \|\mathcal{W}_-\|^2\} dv \\ &= \frac{1}{4\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{96} + \frac{G}{8} \right\} dv \\ &= \frac{1}{32\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{12} + G \right\} dv.\end{aligned}\quad (4.3)$$

From (3.2) and (3.3), taking account of (4.2), we have

$$\begin{aligned}\|R\|^2 &= \frac{1}{24} \{9(\tau^*)^2 - 6\tau\tau^* - 31\tau^2\} + \sum_{i < j} (\rho_{ii} + \rho_{jj})^2 + 4 \sum_{i < j} \rho_{ij}^2 \\ &\quad + 2 \{(\rho_{13}^* - \rho_{31}^*)^2 + (\rho_{14}^* - \rho_{41}^*)^2\} \\ &= \frac{1}{24} \{9(\tau^*)^2 - 6\tau\tau^* - 31\tau^2\} + 2\|\rho\|^2 + \tau^2 \\ &\quad + 2 \{(\rho_{13}^* - \rho_{31}^*)^2 + (\rho_{14}^* - \rho_{41}^*)^2\} \\ &= \frac{1}{24} (3\tau^* - \tau)^2 - \frac{4}{3}\tau^2 + 2 \left( \left\| \rho - \frac{\tau}{4}g \right\|^2 + \frac{\tau^2}{4} \right) + \tau^2 + \frac{1}{2}G \\ &= \frac{1}{24} (3\tau^* - \tau)^2 + 2 \left\| \rho - \frac{\tau}{4}g \right\|^2 + \frac{\tau^2}{6} + \frac{1}{2}G.\end{aligned}\quad (4.4)$$

From (4.4), the Euler number is given by

$$\begin{aligned}\chi(M) &= \frac{1}{32\pi^2} \int_M \{\|R\|^2 - 4\|\rho\|^2 + \tau^2\} dv \\ &= \frac{1}{32\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{24} - 2 \left\| \rho - \frac{\tau}{4}g \right\|^2 + \frac{\tau^2}{6} + \frac{1}{2}G \right\} dv.\end{aligned}\quad (4.5)$$

From (4.3) and (4.5), by Wu's theorem [21], the first Chern number is given by

$$\begin{aligned} c_1(M)^2 &= p_1(M) + 2\chi(M) \\ &= \frac{1}{32\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{6} - 4 \left\| \rho - \frac{\tau}{4}g \right\|^2 + \frac{\tau^2}{3} + 2G \right\} dv. \end{aligned} \quad (4.6)$$

From (4.3) and (4.6), we have the following results.

**Theorem 4.1** *Let  $M = (M, J, g)$  be a compact Tricerri-Vanhecke Bochner flat almost Hermitian Einstein surface. If the first Pontrjagin number  $p_1(M)$  of  $M$  vanishes, then  $M$  is a space of constant sectional curvature  $\frac{\tau}{12}$  ( $\tau \leq 0$ ).*

**Proof** From (4.3), we have  $3\tau^* - \tau = 0$  and  $G = 0$  which implies  $\rho^*$  is symmetric. Thus, from Corollary 3.4,  $M$  is conformally flat and hence,  $M$  is a space of constant sectional curvature  $\frac{\tau}{12}$  since  $M$  is Einstein. It is well-known that a four-dimensional sphere  $S^4$  can not admit an almost complex structure. Therefore, it follows that  $\tau \leq 0$ .  $\square$

**Theorem 4.2** *Let  $M = (M, J, g)$  be a compact Tricerri-Vanhecke Bochner flat almost Hermitian Einstein surface. If the first Chern number  $c_1(M)^2$  of  $M$  vanishes, then  $M$  is locally flat.*

**Proof** Since  $M$  is Einstein, from (4.6), we have

$$c_1(M)^2 = \frac{1}{32\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{6} + \frac{\tau^2}{3} + 2G \right\} dv. \quad (4.7)$$

So, we have

$$\tau = 0, \quad G = 0, \quad 3\tau^* - \tau = 0 \quad (\text{hence} \quad \tau^* = 0). \quad (4.8)$$

Therefore, from (3.1),  $M$  is locally flat.  $\square$

## 5 Compact Tricerri-Vanhecke Bochner flat Kähler Surfaces

We give another proof of the result by Kamishima [9] for the real four dimensional case. Let  $M = (M, J, g)$  be a compact Tricerri-Vanhecke Bochner flat Kähler surface. First, we recall so-called Miyaoka-Yau's inequality [15] :

$$c_1(M)^2 \leq \text{Max} \{2\chi(M), 3\chi(M)\}. \quad (5.1)$$

Since  $M$  is Kähler, the integral formulas (4.3) and (4.5) imply

$$\begin{aligned}\chi(M) &= \frac{1}{32\pi^2} \int_M \left\{ \|R\|^2 - 4\|\rho\|^2 + \tau^2 \right\} dv \\ &= \frac{1}{32\pi^2} \int_M \left\{ \frac{(3\tau^* - \tau)^2}{24} - 2\left\| \rho - \frac{\tau}{4}g \right\|^2 + \frac{\tau^2}{6} + \frac{1}{2}G \right\} dv \\ &= \frac{1}{32\pi^2} \int_M \left\{ \frac{\tau^2}{3} - 2\left\| \rho - \frac{\tau}{4}g \right\|^2 \right\} dv,\end{aligned}\tag{5.2}$$

$$c_1(M)^2 = \frac{1}{32\pi^2} \int_M \left\{ \tau^2 - 4\left\| \rho - \frac{\tau}{4}g \right\|^2 \right\} dv,\tag{5.3}$$

respectively. We now assume that  $\chi(M) \geq 0$ . Then, Miyaoka-Yau's inequality implies

$$c_1(M)^2 \leqq 3\chi(M).\tag{5.4}$$

Then, by (5.2), (5.3), and (5.4) we have

$$\int_M 2\left\| \rho - \frac{\tau}{4}g \right\|^2 dv \leqq 0\tag{5.5}$$

and hence,  $M$  is Einstein (and therefore, in particular, the scalar curvature of  $M$  is constant). Thus, by Theorem 3.5 and (3.1) we see that  $M$  is locally a complex space form. Next, we assume that  $\chi(M) < 0$ . Then, Miyaoka-Yau's inequality implies

$$c_1(M)^2 \leqq 2\chi(M).\tag{5.6}$$

Thus, in this case, by (5.2), (5.3) and (5.6), we have

$$\int_M \frac{\tau^2}{3} dv \leqq 0\tag{5.7}$$

and hence  $\tau \equiv 0$  on  $M$ . Thus, by Theorem 3.5, we see also that  $M$  is locally a product of two oriented surfaces of constant Gaussian curvatures  $K$  and  $-K$  ( $K \neq 0$ ). Summing up the above arguments, we have the following theorem.

**Theorem 5.1** *Let  $M = (M, J, g)$  be a compact Tricerri-Vanhecke Bochner flat Kähler surface. Then  $M$  is locally a complex space form of complex dimension 2, or locally a product of two oriented surfaces of different constant Gaussian curvatures  $K$  and  $-K$  ( $K \neq 0$ ).*

**Remark** The above Theorem 5.1 is included in the result by Y. Kamishima [9] and the proof was first given by B. Y. Chen [4]. We refer to [8, 3, 9] for a further discussion of the Bochner-Kähler manifold. We may note that our proof is different from theirs.

## 6 Tricerri-Vanhecke Bochner flat almost Kähler Einstein surfaces

Let  $M = (M, J, g)$  be a compact Tricerri-Vanhecke Bochner flat almost Kähler Einstein surface. From (3.1), we have

$$\begin{aligned} R_{1313} &= \frac{3\tau^* - 5\tau}{48}, & R_{1324} &= -\frac{3\tau^* - \tau}{48}, & R_{1414} &= \frac{3\tau^* - 5\tau}{48}, \\ R_{1423} &= \frac{3\tau^* - \tau}{48}, & R_{1314} &= 0, & R_{1323} &= 0. \end{aligned} \quad (6.1)$$

From (6.1), we thus have

$$\begin{aligned} u &= -R_{1313} + R_{1324} = -\frac{\tau^* - \tau}{8}, \\ v &= -R_{1414} - R_{1423} = -\frac{\tau^* - \tau}{8}, \\ w &= -R_{1314} - R_{1323} = 0, \end{aligned}$$

and

$$h \equiv (u - v)^2 - 4w^2 = 0. \quad (6.2)$$

which implies that  $M$  is an almost Kähler Einstein surface with Hermitian Weyl tensor [16]. Therefore, by virtue of [1], we see immediately that  $M$  is a Kähler surface. Therefore, taking account of Theorem 3.5, we have the following theorem concerning the Goldberg conjecture [5, 17].

**Theorem 6.1** *Let  $M = (M, J, g)$  be a compact Tricerri-Vanhecke Bochner flat almost Kähler Einstein surface. Then  $M$  is locally a complex space form of complex dimension 2.*

## 7 Remarks

T. Koda [10] has proved that a self-dual almost Hermitian Einstein surface is a space of pointwise constant holomorphic sectional curvature. Further, T. Koda and fourth author of the present paper have proved that a compact self-dual Hermitian Einstein surface is a complex space form of complex dimension 2 [11]. Therefore, we see that a compact Tricerri-Vanhecke Bochner flat Hermitian Einstein surface is a complex space form of complex dimension 2. We herewith introduce an example of a non-compact Tricerri-Vanhecke Bochner flat Hermitian surface of pointwise constant holomorphic sectional curvature which is weakly  $*$ -Einstein but not Einstein.

**Example 4** Let  $\mathbb{C}$  be the set of complex numbers and  $f$  be a non-constant holomorphic function on  $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ . We set  $M = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid \text{Re}f(z) > -1\}$  and assume that  $M$  is nonempty. Further, we set  $u(z) = \text{Re}f(z)$  and  $\sigma(z) \equiv -\log(1 + u(z))$ . Then  $\sigma$  is regarded as a smooth function on  $M$ . Let  $g$  be the canonical Euclidean metric on  $\mathbb{C}^2$  and  $J$  be the complex structure on  $M$  induced by the canonical complex structure on  $\mathbb{C}^2$ . Let  $\bar{g}$  be the Riemannian metric on  $M$  defined by

$$\bar{g} = e^{2\sigma} g = \frac{1}{(1 + u(z))^2} g. \quad (7.1)$$

Since  $M = (M, J, g)$  is a locally flat Hermitian surface (and hence,  $M$  is a Tricerri-Vanhecke Bochner flat Hermitian surface). Since the Tricerri-Vanhecke Bochner curvature tensor  $B(R)$  is conformally invariant,  $(M, J, \bar{g})$  is also Tricerri-Vanhecke Bochner flat. Further, Tricerri and Vanhecke proved that  $(M, J, \bar{g})$  is a space of pointwise constant holomorphic sectional curvature  $c = -e^{2\sigma} \|\text{grad}\sigma\|_g^2$  and  $\tau^* = 4c$ , where  $\|\cdot\|_g^2$  denotes the square norm with respect to the flat metric  $g$  on  $M$  [7]. We may also check that Example 4 is a weakly  $*$ -Einstein manifold.

It is known that a Tricerri-Vanhecke Bochner flat almost Hermitian manifold  $M = (M, J, g)$  is a general complex space form if and only if  $M$  is Einstein and weakly  $*$ -Einstein, and further that a general complex space form of dimension  $2n (\geq 6)$  is locally a complex space form [20]. Concerning this result, Lemence proved that a compact generalized complex space form of dimension four is locally a complex space form of complex dimension 2 [12].

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